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# On the Eleventh Dimension of String Theory

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## Abstract

An explanation of the origin of the hidden eleventh dimension in string theory is given. It is shown that any two sigma models describing the propagation of string backgrounds are related to each other by a Weyl transformation of the world-sheet metric. To avoid this ambiguity in defining two-dimensional sigma models, extra fields are needed. An interesting connection is established with Abelian T-duality.

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# 1. Introduction

There is by now an overwhelming evidence for a unifying origin of all string theories. This "grand unified theory", known as M-theory or F-theory, is supposed to describe in a unified manner the non-perturbative régime of all five superstring theories (See refs.[1, 2, 3] for some reviews and references therein). It seems, therefore, that the five different theories are just five different perturbative manifestations of the same underlying theory [4]. The emergence of M-theory or F-theory was inevitable after the discovery of S-duality in superstrings [5, 6]. This duality, which is the equivalent of electromagnetic duality in Yang-Mills theories, relates two, à priori, different superstring theories [4, 7].

The curious point about M-theory is that it lives in eleven dimensions (or in twelve dimensions in the case of F-theory). The various superstring theories are then obtained by a variety of compactification procedures down to lower dimensions. Given their common origin, one naturally expects to find relations (usually refered to as duality transformations) between the resulting theories. There are mathematical justifications for the apppearence of extra dimensions beyond ten; the critical dimension of superstrings. Indeed, eleven dimensions is the maximum spacetime dimension in which a consistent supersymmetric theory, containing no massless particles with spins greater than two, can be constructed [8, 9]. It is also the dimension, with one time direction, in which supersymmetric extended object (super $p$-branes) are naturally embedded [10, 11, 12]. Allowing for more than one time direction is also another way of building super $p$-branes in more than eleven dimensions [13]. The latter construction is the essence of F-theory [14].

It seems, therefore, that there are hidden dimensions in superstring theories. Their origin is still, however, unclear. It is easier to imagine a lower dimensional theory as descending from a higher dimensional one but the reverse process is much less convincing. An early attempt to explain this hidden dimension was put forward in [15, 16, 17, 18]. It relies on dualizing a vector field into a scalar on the world-volume of a supermembrane propagating in ten dimensions. This method icreases the number of scalars by one. We give, in this note, a natural explanation of the origin of the hidden dimensions in superstrings. We examin this issue at the level of the two-dimensional sigma model and rely on a crucial property of these models, namely their conformal invariance.

A remarkable fact about M-theory is that it does not contain a dilaton field and treats equally all the massless modes of string theory [4]. The dilaton field appears, as a component of the eleven-dimensional metric, only after dimensional reduction [4, 19]. The expectation value of the dilaton field (and the other moduli of the compactification) plays the rôle of the

small parameter of perturbation theory in ordinary string theories. There are therefore no small parameters in M-theory and hence the non-perturbative aspect of this theory.

In contrast, the dilaton field in a non-linear sigma model is treated in a special manner. It is its coupling to the geometry of the two-dimensional world-sheet which distinguishes it from the rest of the massless modes. This coupling is usually given in the form [20]

$$\int \mathrm{d}^2x\sqrt{\gamma}\Phi R^{(2)} \quad , \tag{1}$$

where $\gamma_{\mu\nu}$ is the world-sheet metric, $\gamma$ is its determinant and $R^{(2)}$ is its corresponding scalar curvature. The coupling of the other massless modes involves the metric $\gamma_{\mu\nu}$ and not its derivatives. Furthermore, the dilaton term in a sigma model breaks conformal invariance at the classical level. This last observation will be crucial to us here. It allows for an ambiguity in defining sigma models in two dimensions. It will be shown that the presence of the dilaton term makes it possible to connect any two models by a simple Weyl transformation of the metric $\gamma_{\mu\nu}$. We will show, in section two, that in order to evade this problem and to preserve conformal invariance at the classical level, further fields must be introduced. These will increase the dimension of the target spacetime. Finally we comment, in section three, on a relation between these extra fields and the fields used in the construction of Abelian T-duality (See [21] for a review on T-duality).

## 2. Weyl Transformations

Our starting point is the two-dimensional theory defined by the action[3]

$$S = \int \mathrm{d}^2x L\left(x\right) = \int \mathrm{d}^2x \left[\mathcal{L}\left(x\right) + \sqrt{\gamma}\Phi R^{(2)}\right] \quad , \tag{2}$$

where $\mathcal{L}$ is a two-dimensional Lagrangian and $\Phi\left(x\right)$ is the dilaton field of a string theory. Let also $\widetilde{S}$ be another two-dimensional action given by

$$\widetilde{S} = \int \mathrm{d}^2x \widetilde{L}\left(x\right) \quad . \tag{3}$$

We would like to explore whether the action $\widetilde{S}$ is in anyway related to the action $S$. We will indeed show that $\widetilde{S}$ can be obtained from $S$ by a Weyl rescaling of the metric $\gamma_{\mu\nu}$. This is due, as we will see, to the presence of the dilaton field and to its particular coupling to the scalar curvature.

[3]We will deal only with bosonic theories here. The supersymmetric case follows in a straightforward manner.

In order to see this, we consider a local Weyl rescaling of the metric

$$\gamma_{\mu\nu} \longrightarrow \exp\left[\sigma\left(x\right)\right]\gamma_{\mu\nu} \ . \tag{4}$$

In $d$-dimensions the transformation of the $d$-dimensional Ricci scalar $R^{(d)}$ is given by[4]

$$R^{(d)} \longrightarrow \exp\left(-\sigma\right)\left[R^{(d)} + \left(1-d\right)\nabla^2\sigma + \frac{1}{4}\left(1-d\right)\left(d-2\right)\gamma^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma\right] \ , \tag{5}$$

where $\nabla^2$ is the usual curved Laplacian constructed from the metric $\gamma_{\mu\nu}$. Specialising to $d=2$ we see that the last term in the transformation of $R^{(d)}$ vanishes and the Weyl transformation of the action $S$ is therefore given by

$$\begin{aligned} S &\longrightarrow \int d^2x\left[\mathcal{L}\left(x\right) + \sqrt{\gamma}\Phi R^{(2)} - \sqrt{\gamma}\Phi\nabla^2\sigma\right] \\ &= \int d^2x\left[L\left(x\right) - \sqrt{\gamma}\Phi\nabla^2\sigma\right] \ . \end{aligned} \tag{6}$$

We assumed, for simplicity, that $\mathcal{L}$ is classically invariant under Weyl rescaling.

Let now $G\left(x,y\right)$ denote the inverse of $\nabla^2$, that is the Green's function defined by

$$\nabla_x^2 G\left(x,y\right) = \frac{1}{\sqrt{\gamma\left(x\right)}}\delta^{(2)}\left(x-y\right) \ , \tag{7}$$

where $\nabla_x^2$ is the Laplacian acting at the point $x$. To get the two-dimensional action $\widetilde{S}$ from $S$ throught a Weyl transformation of the form (4), it is sufficient to choose the scale factor $\sigma$ as follows

$$\sigma\left(x\right) = \int d^2y G\left(x,y\right)\frac{1}{\Phi\left(y\right)}\left[L\left(y\right) - \widetilde{L}\left(y\right)\right] \ . \tag{8}$$

Therefore, as long as the dilaton field is different from zero, any two-dimensional theory $\widetilde{S}$ can be obtained from $S$ by a Weyl transformation of the world-sheet metric $\gamma_{\mu\nu}$.

In the case when $S$ is a two-dimensional non-linear sigma model describing the propagation of string massless modes, we have

$$\int d^2x\mathcal{L}\left(x\right) = \int d^2x\left[\sqrt{\gamma}\gamma^{\mu\nu}G_{ij}\left(X\right)\partial_\mu X^i\partial_\nu X^j + \epsilon^{\mu\nu}B_{ij}\left(X\right)\partial_\mu X^i\partial_\nu X^j\right] \ , \tag{9}$$

where $G_{ij}$ and $B_{ij}$, $(i,j=1,\ldots,D)$, are the target space metric and the antisymmetric tensor field respectively. This Lagrangian is indeed classically invariant under a Weyl transformation. Therefore any duality transformation relating two different string backgrounds can be understood as a consequence of a Weyl transformation relating their corresponding sigma models.

[4]Our conventions are such that $R^\mu_{\nu\rho\sigma} = \partial_\rho\Gamma^\mu_{\nu\sigma} + \Gamma^\mu_{\rho\alpha}\Gamma^\alpha_{\nu\sigma} - (\rho\leftrightarrow\sigma)$ and $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$.

One way of avoiding the ambiguity in defining sigma models is to consider, instead of $S$, the following modified action

$$S_{\mathrm{mod}} = \int \mathrm{d}^2 x \left[ \mathcal{L}\left(x\right) + \sqrt{\gamma}\Phi R^{(2)} + a\sqrt{\gamma}\gamma^{\mu\nu}\Phi\partial_\mu A_\nu + b\sqrt{\gamma}\Phi\nabla^2 Y \right] \quad , \tag{10}$$

where $\mathcal{L}$ is the sigma model Lagrangian in (9) and $Y$ and $A_\mu$ are two new fields transforming as

$$A_\mu \longrightarrow A_\mu - \partial_\mu\sigma \quad , \quad Y \longrightarrow Y + \sigma \quad . \tag{11}$$

These last transformations cancel the Weyl transformation of the dilaton term and renders the action classically invariant under Weyl rescaling provided that the two constants $a$ and $b$ satisfy $b - a = 1$.

However, these new fields will impose, at the classical level, a strong constraint on the dilaton field namely, $\partial_i\Phi\left(X\right) = 0$. In order to have a general dilaton field we add to our modified action the following general invariant Lagrangian

$$\begin{aligned} S_{\mathrm{add}} &= \int \mathrm{d}^2 x \left\{ \sqrt{\gamma}\gamma^{\mu\nu} \left[ H\left(X\right) D_\mu Y D_\nu Y + P_i\left(X\right) \partial_\mu X^i D_\nu Y \right] \right. \\ &+ \left. \epsilon^{\mu\nu} \left[ Q_i\left(X\right) \partial_\mu X^i D_\nu Y + N\left(X\right) \partial_\mu A_\nu \right] \right\} \quad , \end{aligned} \tag{12}$$

where we have defined the invariant covariant derivative $D_\mu = \partial_\mu Y + A_\mu$. Therefore the non-linear sigma model that one should start with is given by

$$I\left(X, Y, A\right) = S_{\mathrm{mod}} + S_{\mathrm{add}} \quad . \tag{13}$$

In this last action the gauge field $A_\mu$ appears at most quadratically and can be eliminated through its equations of motion. This procedure leads to a sigma model with $D+1$ scalar fields. Therefore the requirement that a sigma model is conformally invariant at the classical level leads to an extension of the dimension of the target space.

Notice that we could have used a scalar field instead of the gauge field $A_\mu$. This is equivalent to choosing $A_\mu = \partial_\mu V$, where $V$ is a scalar field transforming as $V \longrightarrow V - \sigma$. The important point here is that one needs at least two fields in order to render the action conformally invariant and at the same time to keep the dilaton unconstrained. We will see in the rest of this note a nice relation between this construction and Abelian T-duality.

# 3. Duality and Classical Conformal Invariance

Consider now a non-linear sigma model whose traget space coordinates are denoted $\phi^a$ with $a = 1, \ldots, D$ and an action given by

$$S\left(\phi\right) = \int \mathrm{d}^2 x \left\{ \sqrt{\gamma}\gamma^{\mu\nu} G_{ab}\left(\phi\right) \partial_\mu\phi^a\partial_\nu\phi^b + \epsilon^{\mu\nu} B_{ab}\left(\phi\right) \partial_\mu\phi^a\partial_\nu\phi^b + \sqrt{\gamma}\Phi\left(\phi\right) R^{(2)} \right\} \quad . \tag{14}$$

As explained above, this action can be related to any other two-dimensional theory by a simple Weyl rescaling of the metric $\gamma_{\mu\nu}$. It is therefore understood that a modification of this action on the steps of (13) is necessary. It will be shown shortly that this ambiguity can be also resolved if the metric $G_{ab}$ has an Abelian isometry. Let us suppose that this is indeed the case. We then split the coordinates $\phi^a$ as $\phi^a = (X^i, Y)$ with $i = 1, \dots, D-1$. We assume that the isometry, in this new coordinate system, acts as a simple translation on the coordinate $Y$ only. The non-linear sigma model is then given by

$$\begin{aligned} S(X,Y) &= \int \mathrm{d}^2x \left\{ \sqrt{\gamma}\gamma^{\mu\nu} \left[ G_{ij}(X)\,\partial_\mu X^i \partial_\nu X^j + H(X)\,\partial_\mu Y \partial_\nu Y + P_i(X)\,\partial_\mu X^i \partial_\nu Y \right] \right. \\ &+ \left. \epsilon^{\mu\nu} \left[ B_{ij}(X)\,\partial_\mu X^i \partial_\nu X^j + Q_i(X)\,\partial_\mu X^i \partial_\nu Y \right] + \sqrt{\gamma}\Phi(X)\, R^{(2)} \right\} \ . \end{aligned} \tag{15}$$

This action is invariant under the global shift $Y \longrightarrow Y + \sigma$. To construct the Abelian T-dual of this action, the global isometry is gauged through the introduction of an invariant covariant derivative $D_\mu Y = \partial_\mu Y + A_\mu$ with the gauge field $A_\mu$ transforming as $A_\mu \longrightarrow A_\mu - \partial_\mu \sigma$. In order to obtain a dual sigma model having the same number of degrees of freedom ($D$ coordinates), the field strength of the gauge field is constrained to vanish [22]. This is achieved by means of a Lagrange multiplier field $Z$. The action which leads to the dual sigma model is therefore written as

$$\begin{aligned} S(X,Y,Z,A) &= \int \mathrm{d}^2x \left\{ \sqrt{\gamma}\gamma^{\mu\nu} \left[ G_{ij}(X)\,\partial_\mu X^i \partial_\nu X^j + H(X)\,D_\mu Y D_\nu Y + P_i(X)\,\partial_\mu X^i D_\nu Y \right] \right. \\ &+ \epsilon^{\mu\nu} \left[ B_{ij}(X)\,\partial_\mu X^i \partial_\nu X^j + Q_i(X)\,\partial_\mu X^i D_\nu Y + Z \partial_\mu A_\nu \right] \\ &+ \left. \sqrt{\gamma}\Phi(X)\, R^{(2)} \right\} \ . \end{aligned} \tag{16}$$

As it is well-known, the integration over the Lagrange multiplier leads to $A_\mu = \partial_\mu V$ and upon replacing in the gauged action (16) we get our original sigma model (15) with the field $Y$ replaced by the shifted field $\widetilde{Y} = Y + V$. On the other hand, keeping the Lagrange multiplier and eliminating the gauge field leads to the dual theory. In doing so, the field $Y$ completely disappears (without any use of gauge-fixing) and the Lagrange multiplier $Z$ plays the rôle of this missing field. Hence the number of fields is the same in the original and in the dual theories.

However, the dual theory obtained in this way may as well be obtained, as explained above, by a simple Weyl rescaling of the world-sheet metric of the action (15). Therefore in order to give a sense (and not a mere Weyl rescaling) to this T-duality, the right action to consider is not $S(X,Y,Z,A)$ but a modified one given by

$$I(X,Y,Z,A) = S(X,Y,Z,A) + \int \mathrm{d}^2x \sqrt{\gamma} \left[ a\gamma^{\mu\nu}\Phi(X)\,\partial_\mu A_\nu + b\Phi(X)\,\nabla^2 Y \right] \ . \tag{17}$$

This action is now invariant, when $b - a = 1$, under the finite local transformations

$$\gamma_{\mu\nu} \longrightarrow \exp(\sigma)\,\gamma_{\mu\nu} \ , \ Y \longrightarrow Y + \sigma \ , \ A_\mu \longrightarrow A_\mu - \partial_\mu \sigma \ . \tag{18}$$

The action (17) is exactly of the form of the action of the previous section given in (13) in which the gauge field is restricted to be pure gauge and the Lagrange multiplier $Z$ is identified with $N(X)$. A classical elimination of the gauge field from the action (17) yields a sigma model defined on a $D+1$ dimensional target space. This is because the field $Y$ does no longer disappear as it happened in the case of the action (16).

# 4. Conclusions

We proved in this paper that a two-dimensional non-linear sigma model with a dilaton field is not uniquely defined. This is mainly due to the breaking of classical conformal invariance by the dilaton coupling. It is shown that any two-dimensional theory is obtainable from a sigma model with a dilaton field through a formal Weyl rescaling of the world-sheet metric.

In order to avoid this ambiguity in defining sigma models, conformal invariance must be preserved at the classical level. This is achieved by the introduction of a scalar and a gauge field (though other choices of fields are not excluded). The introduction of these fields increases the dimension of the target spacetime. At first sight the choice of these two fields seems arbitrary. However, their physical interpretation is much more natural in the context of Abelian T-duality.

What remains to be explored here are the renormalisation properties of the modified non-linear sigma model constructed in this note. The requirement that conformal invariance holds at the quantum level leads to imposing some constraints on the string backgrounds. These conditions are in turn derived as equations of motion of a target space action which is the low energy theory of M-theory. Indeed, imposing conformal invariance on the partition function of the action $I(X,Y,Z,A)$ leads to the equation

$$\langle T^\mu_\mu + \partial_\mu J^\mu + \frac{\delta I}{\delta Y} \rangle = 0 \ . \tag{19}$$

The first term is the trace of the energy-momentum tensor defined as $T_{\mu\nu} = \frac{1}{\sqrt{\gamma}}\frac{\delta I}{\delta \gamma_{\mu\nu}}$. In the absence of $Y$ and $A_\mu$ this term usualy leads to the vanishing of the beta-functions. The second term is the current corresponding to the gauge field and is given by $J^\mu = \frac{\delta I}{\delta A_\mu}$. It is clear that in the presence of the fields $Y$ and $A_\mu$, the beta-functions must be modified. This modification depends also on whether the two fields $Y$ and $A_\mu$ are treated as quantum or as background fields. The quantum treatment of the model constructed in this note and other related topics will be considered elsewhere.

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